Jets and Energy-Energy Correlation in QCD

Part II: EEC in $e^+e^-$ Annihilation

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LD, Luo, Shtabovenko, Yang, Zhu, 1801.03219
LD, Moult, Zhu, 1904.nnnnn
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The EEC

• Energy-energy correlation (EEC) in e⁺e⁻ annihilation: one of first infrared safe event-shapes defined in QCD, 40 years ago

\[ \frac{d\Sigma}{d \cos \chi} = \sum_{\text{partons } i,j} \int d\sigma \frac{E_i E_j}{Q^2} \delta(\cos \theta_{ij} - \cos \chi) \]

Collinear parton splitting

\[ E_i \to x E_i + (1 - x) E_i \]

preserves observable. So does soft emission.

Data from wide range of CM energies ➔

\( Q \approx 22 \, \text{GeV} \)  \( Q = 91 \, \text{GeV} \)
Evolution with energy clearly visible

data reviewed in Kardos et al, 1804.09146
Why the EEC?

- Many event-shape variables to choose from: thrust, oblateness, C parameter, heavy jet mass, angularity, jet rates, …
- EEC among the simplest analytically
- Angle $\chi$ lives on a compact domain, $[0, \pi]$: large logarithms on both ends can be resummed
- As $\chi \to 0$, probe jet substructure. Can generalize to computable LHC jet substructure variables, correlating multiple small angles
- Gravitons couple to energy, so AdS/CFT holography can be used to compute at strong gauge coupling (in planar N=4 SYM, not QCD)

Moult, Necib, Thaler, 1609.07483

Hofman, Maldacena, 0803.1467
Numerical results

- EEC computed at NLO numerically in 1980s and 1990s

- Computed numerically at NNLO only 3 years ago
  Del Duca, Duhr, Kardos, Somogyi, Trocsanyi, 1603.08927

- Can now compute analytically at NLO in QCD
Why analytic?

- Validate accuracy of numerical QCD results.
- Compare with analytic NLO result in $N=4$ SYM
  Belitsky, Hohenegger, Korchemsky, Sokatchev, Zhiboedov,
  1309.0769, 1309.1424, 1311.6800

- Study limits as $\chi \to 0, \pi$ to aid resummation of large logarithms there.
LO EEC for $0 < \chi < \pi$ is $O(\alpha_s)$

$$\alpha_s = \frac{g_s^2}{4\pi}$$
How to compute at NLO?

- Use interference method with Feynman diagrams
- **Reverse unitarity**: Treat all momenta as loop momenta, put all cut momenta on shell and impose $\delta(\cos \theta_{ij} - \cos \chi)$
- IBPs/Laporta algorithm Chetyrkin, Tkachov (1981), Laporta (2001)
- Differential equations for master integrals Gehrmann, Remiddi (2000)
  can all be solved in terms of polylogarithms
Structure of QCD result

\[ \frac{1}{\sigma_0} \frac{d\Sigma}{d \cos \chi} = \frac{\alpha_s(\mu)}{2\pi} A(z) + \left( \frac{\alpha_s(\mu)}{2\pi} \right)^2 \left( \beta_0 \log \frac{\mu}{Q} A(z) + B(z) \right) + \mathcal{O}(\alpha_s^3) \]

\[ z = \frac{1}{2}(1 - \cos \chi) \in [0, 1] \]

LO result fits on one line: Basham, Brown, Love, S. Ellis, 1978

\[ A(z) = C_F \frac{3 - 2z}{4(1 - z)^2} \left[ 3z(2 - 3z) + 2(2z^2 - 6z + 3) \ln(1 - z) \right] \]

NLO result will be expressed in terms of classical polylogarithms:

\[ \text{Li}_n(u) = \int_0^u \frac{dt}{t} \text{Li}_{n-1}(t), \quad \text{Li}_1(t) = -\ln(1 - t) \]
Color structure of NLO QCD result

\[ B(z) = C_F^2 B_{lc}(z) + C_F(C_A - 2C_F) B_{nlc}(z) + C_F^2 N_f T_f B_{N_f}(z) \]
Leading color coefficient fits on one page

\[
B_{lc} = \frac{122400z^7 - 244800z^6 + 157060z^5 - 31000z^4 + 2064z^3 + 72305z^2 - 143577z + 63298}{1440(1 - z)z^4} \left( 1 + \frac{244800z^8 - 550800z^7 + 422480z^6 - 126900z^5 + 13052z^4 - 336z^3 + 17261z^2 - 38295z + 19938}{720(1 - z)z^5} \right) \left( 1 + \frac{4z^7 + 10z^6 - 17z^5 + 25z^4 - 96z^3 + 296z^2 - 211z + 87}{24(1 - z)z^5} \right) \left( 1 + \frac{-40800z^8 + 61200z^7 - 28480z^6 + 4040z^5 - 320z^4 - 160z^3 + 1126z^2 - 4726z + 3323}{120z^5} \right) \left( 1 + \frac{1 - 11z^7}{48z^{7/2}} \right) \left( 1 + \frac{120z^6 + 60z^5 + 160z^4 - 2246z^3 + 8812z^2 - 10159z + 4193}{120(1 - z)z^5} \right) \left( 1 + \frac{-2(85z^4 - 170z^3 + 116z^2 - 31z + 3)}{6(1 - z)z^5} \right) \left( 1 + \frac{z^2 + 1}{12(1 - z)} \right) \right),
\]

where

\[
g_1^{(1)} = \log(1 - z), \quad g_2^{(1)} = \log(z), \quad g_1^{(2)} = 2(Li_2(z) + \zeta_2) + \log^2(1 - z),
\]

\[
g_2^{(2)} = Li_2(1 - z) - Li_2(z), \quad g_3^{(2)} = -2Li_2(-\sqrt{z}) + 2Li_2(\sqrt{z}) + \log\left(\frac{1 - \sqrt{z}}{1 + \sqrt{z}}\right)\log(z), \quad g_4^{(2)} = \zeta_2
\]

\[
g_1^{(3)} = -6\left[Li_3\left(-\frac{z}{1 - z}\right) - \zeta_3\right] - \log\left(\frac{z}{1 - z}\right)\left(2(Li_2(z) + \zeta_2) + \log^2(1 - z)\right),
\]

\[
g_2^{(3)} = -12\left[Li_3(z) + Li_3\left(-\frac{z}{1 - z}\right)\right] + 6Li_2(z)\log(1 - z) + \log^3(1 - z),
\]

\[
g_3^{(3)} = 6\log(1 - z)\left(Li_2(z) - \zeta_2\right) - 12Li_3(z) + \log^3(1 - z).
\]
Observations

• Other QCD color coefficients similar in complexity

• See 1801.03219 or https://www.youtube.com/watch?v=WVC1ygsjZNc

• Around both $z = 0$ and $z = 1$, expansion is in integer powers of $z$ (and $\ln z$ or $\ln(1-z)$)

• Individual real/virtual terms have polylog argument $\frac{i\sqrt{z}}{\sqrt{1-z}}$

• Rational function prefactors have no singularities at spurious locations, but their singularities at $z = 0, 1, \infty$ are “too strong” and cancel among different terms

• Similar properties for “Higgs EEC” Luo, Shtabovenko, Yang, Zhu, 1903.07277

• N=4 SYM result (next page) is considerably simpler than QCD, but mainly in rational function prefactors, not transcendental functions
EEC for N=4 SYM at NLO

Belitsky, Hohenegger, Korchemsky, Sokatchev, Zhiboedov, 1311.6800

- Correlator is for scalar source instead of electromagnetic current (but the precise source doesn’t matter much)

\[ F(z; a) = a F_1(z) + a^2 \left[ (1 - z) F_2(z) + F_3(z) \right] \]

where

\[ a = g_{YM}^2 N / (4\pi^2) \]

\[ F_1(z) = -\ln(1 - z) \]

\[ F_2(z) = 4\sqrt{z} \left( \text{Li}_2(-\sqrt{z}) - \text{Li}_2(\sqrt{z}) + \frac{\ln z}{2} \ln \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right) + (1 + z) \left[ 2\text{Li}_2(z) + \ln^2(1 - z) \right] + 2 \ln(1 - z) \ln \left( \frac{z}{1 - z} \right) + z \frac{\pi^2}{3} \]

\[ F_3(z) = \frac{1}{4} \left\{ (1 - z)(1 + 2z) \left[ \ln^2 \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \ln \left( \frac{1 - z}{z} \right) - 8\text{Li}_3 \left( \frac{\sqrt{z}}{\sqrt{z} - 1} \right) - 8\text{Li}_3 \left( \frac{\sqrt{z}}{\sqrt{z} + 1} \right) \right] - 4(z - 4)\text{Li}_3(z) + 6(3 + 3z - 4z^2)\text{Li}_3 \left( \frac{z}{z - 1} \right) - 2z(1 + 4z)\zeta_3 + 2 \left[ 2(2z^2 - z - 2) \ln(1 - z) + (3 - 4z)z \ln z \right] \text{Li}_2(z) + \frac{1}{3} \ln^2(1 - z) \left[ 4(3z^2 - 2z - 1) \ln(1 - z) + 3(3 - 4z)z \ln z \right] + \frac{\pi^2}{3} \left[ 2z^2 \ln z - (2z^2 + z - 2) \ln(1 - z) \right] \right\} \]

- No uniform or maximal transcendentality principle – except for $\chi \rightarrow \pi$
Belitsky et al. method for N=4 SYM

- Very different from “QCD method”, which uses dimensional regularization; divergences cancel between virtual and real
- Exploit conformal invariance of 4-point function with two “energy flow operators”

\[
\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\rangle_q = \int d^4x e^{iq\cdot x} \langle 0|O^\dagger(x)\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)O(0)|0 \rangle
\]

\[
\mathcal{E}(\vec{n}) = \int_{-\infty}^{\infty} d\tau \lim_{r\to\infty} r^2 n^iT_{0i}(t = \tau + r, r\vec{n})
\]

- Analytically continue from Euclidean to physical region using double Mellin transform
- No infrared divergences at any step!
- Recently pushed to NNLO (semi-analytic): Henn, Sokatchev, Yan and Zhiboedov, 1903.05314
Analytic properties of QCD moments

• With analytic formulae, compute the integrals

\[ B_N = \int_0^1 dz \, z^N B(z) \]

numerically to high accuracy, for each color coefficient

• Using PSLQ, it is always of the form

\[ B_N = r_N^{(4)} \zeta(4) + r_N^{(3)} \zeta(3) + r_N^{(2)} \zeta(2) + r_N^{(0)} \]

where the \( r_N^{(w)} \) are rational numbers.

• E.g.

\[ B_3(C_A) = -\frac{207}{2} \zeta(4) + \frac{14902}{35} \zeta(3) - \frac{553}{450} \zeta(2) - \frac{2369041}{5040} \]

• Could they be zeta values at higher loop orders too?

• Expression for general \( N \) in terms of \( \psi(N) \) functions?
Fixed order QCD vs. Z pole data

Tulipant, Kardos, Somogyi, 1708.04093

\[ z \to 1 \]

\[ z \to 0 \]
To measure strong coupling $\alpha_s$:

- Add NNLL $z \to 1$ resummation
- + MC estimate of nonperturbative contributions

Kardos, Kluth, Somogyi, Tulipant, Verbytskyi, 1804.09146

$$\alpha_s(M_Z) = 0.11750 \pm 0.00018(exp.) \pm 0.00102(hadr.)$$
$$\pm 0.00257(ren.) \pm 0.00078(res.)$$

Competitive measurement of $\alpha_s$

Still room for theory improvement:
- $\to$ NNNLO (approx.?)
- + NNNLL $z \to 1$ resummation
- + (N?)NLL $z \to 0$ resummation
Back-to-back limit, $z \to 1$

$$B(z) = C_F \left\{ \frac{1}{1-z} \left[ \frac{1}{2} C_F \ln^3(1-z) + \ln^2(1-z) \left( \frac{11C_A}{12} + \frac{9C_F}{4} - \frac{N_fT_f}{3} \right) ight. ight.$$ 

$$+ \ln(1-z) \left( C_A \left( \frac{\zeta_2}{2} - \frac{35}{72} \right) + C_F \left( \zeta_2 + \frac{17}{4} \right) + \frac{N_fT_f}{18} \right) 
+ C_A \left( \frac{11\zeta_2}{4} + \frac{3\zeta_3}{2} - \frac{35}{16} \right) 
+ C_F \left( 3\zeta_2 - \zeta_3 + \frac{45}{16} \right) + N_fT_f \left( \frac{3}{4} - \zeta_2 \right) \right]$$

$$+ \left( \frac{C_A}{2} + C_F \right) \ln^3(1-z) + \ln^2(1-z) \left( \frac{27C_A}{8} + \frac{13C_F}{2} - \frac{N_fT_f}{2} \right) 
+ \ln(1-z) \left[ C_A \left( 22\zeta_2 - \frac{2011}{72} \right) + C_F(47 - 19\zeta_2) + N_fT_f \left( \frac{361}{36} - 4\zeta_2 \right) \right]$$

$$+ C_A \left( \frac{6347\zeta_2}{80} - 21\zeta_2 \ln 2 - \frac{137\zeta_3}{4} - \frac{3305}{72} \right) 
+ C_F \left( -\frac{1727\zeta_2}{20} + 42\zeta_2 \ln 2 + \frac{121\zeta_3}{2} + \frac{3437}{96} \right) 
+ N_fT_f \left( -\frac{1747\zeta_2}{120} + 12\zeta_3 + \frac{2099}{144} \right) \right\} + \mathcal{O}(1-z)$$

- Double log behavior, $\ln^{2L+1}(1-z)/(1-z)$ characteristic of Sudakov suppression from soft/collinear gluon emission. Collins, Soper, …
- Coefficients of leading-power terms agree precisely with NNLL resummation DeFlorian, Grazzini, hep-ph/0407241
\[ z \rightarrow 1 \] (cont.)

- Factorization theorem recently proved: Relate EEC to back-to-back production of identified hadrons\footnote{Collins, Soper 1981-1982}
  - Should allow NNNLL resummation soon

Soft gluons contribute, but only via \textit{recoil}, by deflecting the hard quark jet

\[ \bar{n}_a \cdot p_a = Q \]
\[ \bar{n}_b \cdot p_b = Q \]
\[ \frac{\pi - \theta_{ij}}{2} \approx \frac{1}{Q} \left| \frac{k_{h,i}^a}{x_i} + \frac{k_{h,j}^b}{x_j} - \vec{k}_{\perp,i}^h \right| \]
Intra-jet limit, $z \rightarrow 0$

\[ B(z) = C_F \left\{ \frac{1}{z} \ln z \left( -\frac{107 C_A}{120} + 25 C_F \frac{1}{32} + \frac{53 N_f T_f}{240} \right) + C_A \left( -\frac{25 \zeta_2}{12} + \frac{\zeta_3}{2} + \frac{17683}{2700} \right) \right. \\
+ C_F \left\{ \frac{43 \zeta_2}{12} - \zeta_3 - \frac{8263}{1728} - \frac{4913 N_f T_f}{3600} \right\} \\
+ \ln z \left[ C_A \left( \frac{33 \zeta_2}{2} - \frac{703439}{25200} \right) + C_F \left( \frac{42109}{1200} - 21 \zeta_2 \right) + N_f T_f \left( \frac{86501}{12600} - 4 \zeta_2 \right) \right] \\
\left. + C_A \left( \frac{213 \zeta_2}{5} - \frac{101 \zeta_3}{2} - \frac{26986007}{5292000} \right) + C_F \left( \frac{1541 \zeta_2}{30} + 65 \zeta_3 + \frac{18563}{2700} \right) \\
+ N_f T_f \left( -\frac{46 \zeta_2}{3} + 12 \zeta_3 + \frac{2987627}{330750} \right) \right\} + O(z) \]

- Single log behavior, $\ln^L z/z$ characteristic of pure collinear observable.
- Leading log (LL) resummation first performed in “jet calculus” approach Konishi, Ukawa, Veneziano, Phys.Lett.1978,1979
- Coefficients of leading-power terms agree precisely with LL result Richards, Stirling, Ellis, NPB229, 317, 1983
$z \to 0$ (cont.)

- Limit dominated by collinear emission. At leading log, only a single moment $N=3$ of time-like splitting function dominates Konishi, Ukawa, Veneziano, Richards, Stirling, Ellis, Hofman, Maldacena, 0803.1467

Energy weighting $\to$

\[
\int_0^1 dx \ x (1-x) \ P_{ij}(x) \to - \int_0^1 dx \ x^2 P_{ij}(x) \equiv \gamma_{ij}^{(N=3)}
\]

Momentum sum rule controls $x^1$ term, $\to$ can drop it.

\[
\int_0^1 dx \ x P_{ij}(x) \equiv -\gamma_{ij}^{(N=2)}
\]
LL resummed formula

Richards, Stirling, Ellis, NPB229, 317, 1983

\[
\frac{1}{\sigma_0} \frac{d\Sigma}{d\cos \chi} = \frac{\alpha_s(\sqrt{z}Q)}{16\pi} \frac{z}{\alpha_s(Q)} \sum_{i,j=q,g} \Gamma_{ij}^{(0)} \left[ \frac{\alpha_s(\sqrt{z}Q)}{\alpha_s(Q)} \right]^{j_q} \Gamma^{(0)} / b_0
\]

\[
\Gamma_{ij}^{(0)} = \begin{bmatrix}
\frac{25}{6} C_F & -\frac{7}{15} N_f \\
-\frac{7}{6} C_F & \frac{14}{5} C_A + \frac{2}{3} N_f
\end{bmatrix}
\]

\[
b_0 = \frac{11}{3} C_A - \frac{2}{3} N_f
\]

One-loop (LO) $N=3$ time-like moments

To expand back into fixed order:

\[
\frac{\alpha_s(\sqrt{z}Q)}{\alpha_s(Q)} = \left[ 1 + b_0 \frac{\alpha_s(Q)}{4\pi} \ln z \right]^{-1}
\]
Beyond LL as $z \to 0$

• Factorize on single parton states, similar to production of identified hadrons $h$ with momentum $p_h = x \times Q/2$

$$
\frac{d\sigma(e^+e^- \to h + X)}{dx} = \sum_{i=q,g} \int_0^1 dx_i \frac{d\sigma(e^+e^- \to i + X)}{dx_i} D_{i \to h}(x/x_i)
$$

..., Mitov, Moch, Vogt, 2006
Moch, Vogt, 0709.3899,
Almasy, Moch, Vogt, 1107.2263

perturbative hard function, computed to NNLO + evolution
nonperturbative fragmentation function
All orders factorization formula

- Cumulant
  \[ \Omega(z, \ln \frac{Q^2}{\mu^2}, \mu) = \int_0^z dz' \Sigma(z', \ln \frac{Q^2}{\mu^2}, \mu) \]

\[ \Omega(z, \ln \frac{Q^2}{\mu^2}, \mu) = \int_0^1 dx x^2 J^T \left( \ln \frac{zx^2Q^2}{\mu^2}, \mu \right) \cdot \vec{H}(x, \ln \frac{Q^2}{\mu^2}, \mu) \]

- Reuses hard function \( H_i = \frac{d\sigma}{dx_i} \)

- Replaces nonperturbative fragmentation function with perturbative jet function \( J \) which includes the small angle EEC measurement.

- Dependence of \( J \) is on its only physical scale, \( zx^2Q^2 = q_T^2 \)
All orders factorization formula

\[ \Omega(z, \ln \frac{Q^2}{\mu^2}, \mu) = \int_0^z dz' \Sigma(z', \ln \frac{Q^2}{\mu^2}, \mu) \]

\[ \Omega(z, \ln \frac{Q^2}{\mu^2}, \mu) = \int_0^1 dx x^2 \tilde{J}^T \left( \ln \frac{zx^2Q^2}{\mu^2}, \mu \right) \cdot \tilde{H}(x, \ln \frac{Q^2}{\mu^2}, \mu) \]

- Computed \( J \) directly to \( O(\alpha_s) \) so far
  \( \rightarrow \) NLL accuracy
- Reproduces coefficient of \( \alpha_s^2 (\ln z)^0/z \)
in fixed order NLO result for both \( e^+e^- \) and Higgs
Evolution of jet function

- Evolution of hard function involves time-like splitting kernel, $P_T(y, \mu)$.
- $\Omega$ is RGE invariant, i.e. independent of $\mu$
- Leads to evolution equation for $J$:

$$\frac{dJ^T}{d \ln \mu^2} \frac{\ln \frac{zQ^2}{\mu^2}, \mu}{dy y^2} = \int_0^1 dy y^2 J^T \left( \ln \frac{zy^2Q^2}{\mu^2}, \mu \right) \cdot P_T(y, \mu)$$

- LL evolution only uses $N=3$ time-like moments $(y^2)$, but beyond LL, need “nearby” moments.
Counting the order

\[
\begin{align*}
\Sigma(z) & : 1 & \alpha_s & \alpha_s^2 & \alpha_s^3 & \alpha_s^4 & \ldots \\
\delta(z) & : & & & & & \\
\end{align*}
\]

- To get to NNLL require:
  - NNLO splitting kernel
    Moch, Vermaseren, Vogt
  - NNLO hard function
    Mitov, Moch, 2006;
    Almasy, Moch, Vogt, 2011
  - NNLO jet function
    Very challenging!
Use “unitarity” to get $\alpha_s^2 \delta(z)$

- Get $\alpha_s^2 \delta(1-z)$ in course of resumming $z \to 1$
- Know $\alpha_s^2$ distribution for $0 < z < 1$, so we can integrate it over this range, up to the delta functions.
- Total cross section

$$\sigma = \int_0^1 dz \frac{d\sigma}{dz}$$

also known, for $e^+e^-$ and Higgs, to very high order, e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt, 1707.01044

- Use the two $\delta(z)$ coefficients to fix 2-loop $J_q, J_g$
NNLO QCD $\alpha_s^3$ coefficient

Del Duca, Duhr, Kardos, Somogyi, Trocsanyi, 1603.08927

rise dominated by NLL term, not LL
$n_f = 5, \mu = Q$

- exact, arXiv:1603.08927, fig. 6
- leading power, from NNLL
Conclusions

- Analytical results possible at NLO in QCD for at least one event shape in $e^+e^-$ annihilation, the EEC
- Transcendental structure no worse than for N=4 SYM, but rational functions considerably more complicated
- Limiting values useful for checking soft-gluon resummation for $z \to 1$ – also beyond leading power when available
  Moult, Stewart, Vita, Zhu, 1804.04665
- Also very useful in developing formalism for collinear resummation, $z \to 0$, now to NNLL
  LD, Moult, Zhu, to appear
- May eventually lead to more precise value of $\alpha_s$, as well as more precise jet substructure understanding at LHC
Extra Slides
Reverse unitarity

Anastasiou, Melnikov, hep-ph/0207004;
Anastasiou, LD, Melnikov, Petriello, hep-ph/0312266

• Phase space integral over final-state partons is like a loop integral with \( \delta(p_i^2) \) factor for every propagator crossing the cut, and with one extra delta function, which can be turned into a fake propagator:

\[
\delta[M_{ij}(\chi)] = \frac{1}{2\pi i} \left[ \frac{1}{M_{ij}(\chi) - i\varepsilon} - \frac{1}{M_{ij}(\chi) + i\varepsilon} \right]
\]

where

\[
M_{ij}(\chi) = (p_i \cdot Q p_j \cdot Q)(\tilde{n}_i \cdot \tilde{n}_j - \cos \chi)
\]

\[
= (p_i \cdot Q p_j \cdot Q)(1 - \cos \chi) - p_i \cdot p_j
\]

• Nonlinear in parton momenta \( p_i, p_j \)

• Sum over \( i,j \)
Integration by parts (IBP)

• Multi-loop integration technology

\[ 0 = \int d^D p d^D q \cdots \frac{\partial}{\partial q^\mu} \frac{k^\mu}{p^2 q^2 (p+q)^2} \cdots \]

Chetyrkin, Tkachov (1981)

• Reduces problem to system of linear equations, initially solved recursively by MINCER, now by Laporta algorithm, in terms of “master integrals”

Gorishnii, Larin, Surguladze, Tkachov (1989)

Laporta, hep-ph/0102033

No-scale problem \( \Rightarrow \) like total hadronic cross section
maximal analytic simplicity:
pure numbers, Riemann zeta values

\[ \zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} \]

\[ \frac{R_{e^+e^-}}{R^{(0)}} = 1 + \frac{\alpha_s}{\pi} + \left( \frac{\alpha_s}{\pi} \right)^2 \left[ -11\zeta(3) + \frac{365}{24} + n_f \left( \frac{2}{3} \zeta(3) - \frac{11}{12} \right) \right] 
+ \left( \frac{\alpha_s}{\pi} \right)^3 \left[ \frac{275}{6} \zeta(5) - \frac{1103}{4} \zeta(3) - \frac{121}{8} \zeta(2) + \frac{87029}{288} \zeta(3) - \frac{11}{18} \zeta(2) + \frac{151}{162} \right] \]

EEC is “next-to-simplest case”
QCD: NLO program EVENT2 validated

M. Seymour